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**APPLICATION OF THE REGULAR REPRESENTATION OF SINGULAR INTEGRALS
TO THE SOLUTION OF THE SECOND FUNDAMENTAL PROBLEM
OF THE THEORY OF ELASTICITY**

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A regular representation is proposed for singular integrals present in integral equations of the second fundamental problem of elasticity theory. This representation is used to realize the successive approximations method in solving internal and external problems. Questions of constructing a computational scheme are discussed.

Use of potential theory apparatus permits reduction of the analysis of the fundamental boundary value problems of elasticity theory to integral equations [1]. To solve the second fundamental problem, Weil constructed regular integral equations of the second kind which generally possess eigenfunctions. Hence, their solution can be realized only after all the eigenfunctions of the adjoint equation have been determined, which is a complicated problem.

The application of a generalized elastic potential of a simple layer also reduces the mentioned boundary value problem to integral equations of the second kind. It is true these equations are not Fredholm equations in the classical form since their kernels have a second order polarity, and the corresponding integrals should be understood in the principal value sense. Consequently, the equations themselves are called singular. The equations mentioned possess quite favorable spectral properties. In the case of the external problem (we denote it by T_n) the equation is solvable for an arbitrary right-hand side. In the case of the internal problem (T_i), the equation is solvable when the right-hand side satisfies definite conditions but they agree with the conditions for existence of the solution of the initial problem of elasticity theory (the principal vector and the principal vector-moment of the external forces equal zero) and hence are assumed satisfied according to the formulation of the problem.

Each of the methods of solving the integral equations starts from the possibility of evaluating the integral terms for some representation of the required density. The associated difficulties are aggravated in solving singular, especially nonuniform, integral equations.

Questions of realizing the mechanical quadrature method in application to singular

integral equations of the fundamental three-dimensional problems of elasticity theory have been examined in [2-4].

A representation is given below for singular integrals present in the equations of the second fundamental problem of elasticity theory, which does not explicitly contain singular terms and removes the difficulties mentioned above for their evaluation.

Let us represent the singular equation of the second fundamental problem as

$$\varphi(p) - \lambda I = f(p), \quad I = \int_S \Gamma_1(p, q) \varphi(q) dS_q \quad (1)$$

The problem T_a corresponds to the value $\lambda = 1$ and the problem T_i to the value $\lambda = -1$. The matrix $\Gamma_1(p, q)$ is obtained as a result of the stress operator acting on the Kelvin-Somigliana tensor, S is the surface bounding the body under consideration and is a Liapunov surface, and the function $f(p)$ agrees with the boundary values of the stress vector in the problem T_i and is equal, but opposite in sign, to it in the problem T_a . This function should belong to the Hölder-Lipschitz class.

It has been proved [5] that the Fredholm alternatives are valid for the singular integral equation (1). It follows from the uniqueness theorem for the solution of the fundamental problems of elasticity theory and the Fredholm alternatives [6] that (1) (considered in the complex λ plane) has just real eigennumbers which are not less than unity in absolute value. The number $\lambda = 1$ is not an eigennumber. Consequently, the problem T_a turns out to be always solvable. The number $\lambda = -1$ is an eigennumber. Since the eigenfunctions of the adjoint equation correspond to displacement of the elastic body as a rigid whole, then the solvability conditions for (1) agree in this case with the existence conditions for the solution of the considered elasticity theory problem and are, hence, considered satisfied.

Let us transform the singular integral in (1) thus:

$$I = -\varphi(p) + \int_S \{\Gamma_1(p, q) \varphi(q) - \Gamma_2(p, q) \varphi(p)\} dS_q \quad (2)$$

$$\int_S \Gamma_2(p, q) dS_q = -E$$

(the matrix $\Gamma_2(p, q)$ is the kernel of a double-layer potential and E is the unit matrix). We consider the integral obtained above. The singular terms in the matrices $\Gamma_1(p, q)$ and $\Gamma_2(p, q)$ agree, and the function $\varphi(p)$ belongs to the Hölder-Lipschitz class (as the solution of (1) under the constraints formulated on the surface and the right-hand side [7]). Hence, the integral mentioned is improper.

Using the identity (2), we can obtain a regular representation of the integral equations (1) themselves and can use known methods for their solution. Application of the method of mechanical quadratures is not expedient since it is required to solve linear systems of very high order even for fairly smooth surfaces and a smoothly varying load. Moreover, the question of the convergence of the approximate solutions to the exact solution remains open.

We turn to successive approximations, whose application to solve the singular integral equations of elasticity theory has been examined in [8]. The solution is sought in the form of the series

$$\varphi(p) = \sum_{n=0}^{\infty} \lambda^n \Psi_n(p) \quad (3)$$

Substituting it into (1) and equating coefficients of identical powers of λ , we arrive at the recurrence relations

$$\Psi_n(p) = \int_S \Gamma_1(p, q) \Psi_{n-1}(q) dS_q, \quad n = 1, 2, \dots, \quad \Psi_0(p) = f(p) \quad (4)$$

The presence of the regular representation (2) permits evaluation of the terms in the series (3) with arbitrary accuracy.

Let us consider the question of the convergence of the series (3) in the problems T_a and T_i . We note that the spectral properties of (1) and of the equation of the Neumann problem (for the Laplace equation) basically agree. The convergence of successive approximations in this latter case has been investigated in [9]. The results obtained have been extended [8] to the integral equations of elasticity theory.

In the case of the problem T_a the series (3) generally diverges since the appropriate value of λ is on the circle of convergence of the resolvent. A convergent representation of the solution is

$$\varphi(p) = \frac{1}{2} \Psi_0(p) + \frac{1}{2} \sum_{n=0}^{\infty} \{\Psi_{n+1}(p) + \Psi_n(p)\}$$

Other convergent representations, obtained by means of analytic continuation in the parameter λ [10], are also possible.

Let us consider the problem T_i . We elucidate the proof of the convergence of the series (3) by following the comment to [8] presented in [7]. Let us recall the expression for the resolvent of (1)

$$R(p, q, \lambda) = \frac{1}{1 + \lambda} \left[\sum_{k=1}^6 b^k(p) \chi^k(q) \right] + A(p, q, \lambda) \quad (5)$$

Here $A(p, q, \lambda)$ is a holomorphic matrix in λ in a circle of radius greater than unity, $b^k(p)$ and $\chi^k(p)$ ($k = 1, 2, \dots, 6$) are biorthonormal systems of eigenfunctions of the initial equation and its adjoint. The solution of (1) is represented by means of the resolvent (5) as

$$\varphi(p) = f(p) + \lambda \int_S A(p, q, \lambda) f(q) dS_q \quad (6)$$

The term corresponding to the first term is missing because of compliance with the condition of orthogonality of the right-hand side to all the functions $\chi^k(q)$

$$\int_S f(q) \chi^k(q) dS_q = 0, \quad k = 1, 2, \dots, 6 \quad (7)$$

Therefore, the required function $\varphi(p)$ turns out to be represented as a holomorphic expression in λ in a circle of radius greater than unity, namely, this expression is constructed by the series (3). Consequently, the series should converge for $\lambda = -1$, despite the presence of a pole of the resolvent.

We note that upon compliance with conditions (7) the series (3) also turns out to be convergent for the problem T_a .

It must be noted that the result of Pham The Lai used above (and the N. M. Giunter proof in the case of the external Neumann problem, equivalently) is valid only for the exact evaluation of the integrals at each iteration. Since the realization of the recurrence relations (4) can be realized only with an error, its influence on the convergence of the algorithm should be clarified.

The question of the solution of integral equations of the second kind (in the spectrum) by successive approximations is considered in [11] from the aspect of the theory of in-

correct problems. It was assumed that all the calculations were carried out absolutely accurately, but the right-hand side of the equation was given with some error (δ). It was proved that the process converges if the product

$$n\delta^2 \rightarrow 0 \quad (8)$$

(where n is the number of the iteration).

As applied to the problem under study, the error in the right-hand side can be treated as the error of quadrature formulas. Hence, it is impossible to keep an arbitrarily large number of terms in (3) (according to (8)) for a fixed partition of the surface into elementary polygons. Instead of constructing the analog to the condition (8), an equivalent algorithm can be proposed: for a fixed number of iterations, the computations are carried out with a successive decrease in the size of the elementary regions, which assures achievement of previously assigned accuracy for a finite sum (3). For an increase in the number of terms in the series it is necessary to introduce a still finer partition in a suitable way. Realization of the proposed algorithm is connected with carrying out a large volume of computations.

Let us consider another (more effective) algorithm. We recall that Pham The Lai proved his theorem by starting from the fact that each function $\Psi_n(p)$ is orthogonal to all the functions $\chi^k(p)$. We use this circumstance for the appropriate correction in the calculation of the functions $\Psi_n(p)$. Initially, let us replace the right-hand side in the equation (for $\lambda = -1$)

$$f^*(p) = f(p) - \sum_{k=1}^6 \chi^k(p) \int_S f(q) \chi^k(q) dS_q \quad (9)$$

We assume that the functions $\chi^k(p)$ are developed in orthonormal form.

For an accurate calculation of the additional terms, they should vanish from conditions (7). Some, generally small, additions are obtained in calculations by any quadrature formula. The function $f^*(p)$ (in contrast to $f(p)$) is strictly orthogonal to each of the functions $\chi^k(p)$ (within the framework of the quadrature formula used if orthonormalization of the functions $\chi^k(p)$ is accomplished by means of the same formula).

In order for any of the interactions be strictly orthogonal to the functions $\chi^k(p)$, a transformation analogous to (9) and of the form

$$\Psi_n^*(p) = \Psi_n(p) - \sum_{k=1}^6 \chi^k(p) \int_S \Psi_n(q) \chi^k(q) dS_q \quad (10)$$

must be accomplished each time.

The functions $\chi^k(p)$ should generally be understood in the discussions presented as eigenfunctions of the adjoint approximate equation which occurs during realization of the computational scheme.

If the shape of the surface and the loading are such that there are three planes of symmetry, and discretization is accomplished corresponding to a symmetric manner, then all the additions vanish automatically.

Let us describe one of the possible schemes for realizing the proposed algorithm to solve the integral equation (1).

We separate the surface S into small polygons whose vertices will be called nodal points and denoted by q_j . We select a point located in the central part in each polygon, at the center of gravity, say, and call them reference points denoting by p_i .

Let us initially determine the values of the function $\Psi_0(p)$ at all the reference and nodal points by equating them to the right-hand sides of the equation. Furthermore, we

find the function $\Psi_1(p)$ at the reference points by using some quadrature formula in the regular representation (2) to calculate the improper integral, by assuming a term of the integral sum, say, to be equal to the product of the mean value with respect to the appropriate nodal points of the expression

$$\Gamma_1(p_i, q_j) \Psi_0(q_j) - \Gamma_2(p_i, q_j) \Psi_0(p_i)$$

on the area of the polygon. We determine the values of $\Psi_1(p)$ at the nodal points by interpolation by starting from the values of the functions $\Psi_1(p)$ at the nearest reference points. The subsequent constructions are obvious.

The external and internal problems of elasticity theory for a sphere were considered in order to determine the efficiency of the approach proposed. The loading was reduced to hydrostatic pressure in both cases (because of the simplicity of determining the exact value of the density required [12]). We assume a unit pressure. Then the exact values of the density (in absolute value) turn out to equal $\Phi_i = \frac{2}{3} (1 - \nu) / (1 + \nu)$ and $\Phi_a = -\frac{2}{3} (1 - \nu) / (1 - 2\nu)$ (ν is the Poisson's ratio).

Let us partition the sphere surface by introducing a geographical coordinate system ($-\pi \leq \varphi \leq \pi$, $-\pi/2 \leq \theta \leq \pi/2$) by dividing the angle φ into n equal parts, and the angle θ into m parts. The coordinates of the reference points were given by the formulas

$$\varphi_i = (2i / (n - 1)) \pi, \quad i = 0, 1, \dots, n; \quad \theta_j = (j / m - 1/2) \pi \\ j = 0, 1, \dots, m$$

Presented in Table 1 are the results of computations for the functions Ψ_n , Φ_a and Φ_i (keeping five terms) at the pole (A) and equator (B). The numbers n and m were taken equal to eight. The Poisson's ratio was assumed to be 0.3. We note that in this case the exact value of Φ_a is -1.3125 while Φ_i is 0.807 (independently of the location of the point, naturally).

Table 1

	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Φ_a	Φ_i
A	1.000	0.247	0.063	0.015	0.004	-1.329	0.804
B	1.000	0.269	0.064	0.016	0.004	-1.353	0.784

The main assertions of the research have been published briefly in abstracts. (Perlin, P. I., On a method of evaluating singular integrals and its application to the solution of singular integral equations of three-dimensional problems of elasticity theory. All-Union School on Theoretical Investigations by the Method of the Mechanics of Continuous Media. Abstracts of Reports, 1973).

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ON THE NATURE OF THE CONTACT STRESS SINGULARITIES UNDER AN ANNULAR STAMP

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The nature of the normal stress singularities under an annular stamp as one approaches the outer and inner contours is clarified.

An approach permitting to obtain an asymptotic expansion for the contact stress which consists of one term (an asymptotic representation in the Erdelyi terminology), is developed. The method proposed permits the investigation of a number of contact problems associated with an annular stamp. However, only an axisymmetric contact problem is considered in this paper. A survey of the research devoted to the problem of impressing an annular stamp into an elastic half-space is presented in [1, 2].

The problem of the behavior of solutions of elasticity theory boundary value problems in the neighborhood of points and lines of separation of boundary conditions was examined in [3 — 8], etc.

1. We use a r, φ, z cylindrical coordinate system, whose z -axis is perpendicular